***Linear Partial Differential Equations***

***Of Order One.***

**Partial Differential Equation:**An equation involving partial derivatives of one or more dependent variables with respect to more than one independent variables is called a partial differential equation.

**Example:** 



**Order and Degree:** The order of the highest ordered derivative involving in a partial differential equation is called the order of the partial differential equation.

Again, the exponent of the highest ordered derivative involving in a partial differential equation is called the degree of the partial differential equation, after freed from radicals and fractions in its derivatives.

**Example:** Consider the following partial differential equation,



The order of this equation is 2 and the degree is 1.

***NOTE:*** In the case of two independent variables, we usually assume that,  where *x* and *y* are independent variables and *z* is the dependent variable. We adopt the following notations throughout the study of partial differential equations.

,  ,  ,  , .

**Formation of Partial Differential Equation:**A partial differential equation can be formed by the elimination of arbitrary constants or arbitrary functions.

**Problem-01:** Find a partial differential equation by eliminating *a* and *b* from the equation .

**Solution:** Given that, 

Differentiating (1) partially with respect to *x*, we get



And differentiating (1) partially with respect to *y*, we get



Substituting these values of *a* and *b* in (1), we get



which is the required partial differential equation.

**Problem-02:** Eliminate arbitrary constants from the equation, .

**Solution:** Given that, 

Differentiating (1) partially with respect to *x*, we get





And differentiating (1) partially with respect to *y*, we get





Adding equations (1) and (2), we get





[using eq.(1)]



which is the required partial differential equation.

**Problem-03:** Find the differential equation of all spheres of radius , having centre in the *xy* - plane.

**Solution:** From the coordinate geometry of three dimensions, the equation of any sphere of radius , having centre (*h, k, 0*)in the *xy*–plane is,



Differentiating (1) partially with respect to *x*, we get







And differentiating (1) partially with respect to *y*, we get







From equations (1), (2) and (3), we get



which is the required partial differential equation.

**Problem-04:** Eliminate the arbitrary constants *a*, *b* and *c* from the relation.

**Solution:** Given that, 

Differentiating (1) partially with respect to *x*, we get





Again, differentiating (2) partially with respect to *x*, we get



From (2), we get



Putting this value of  in (4), we obtain









which is the required partial differential equation.

**Problem-05:** Form partial differential equation by eliminating constant *A* and *p* from .

**Solution:** Given that, 

Differentiating (1) partially with respect to *x*, we get



Again, differentiating (2) partially with respect to *x*, we get



Similarly, differentiating (1) partially with respect to *t*, we get



Again, differentiating (4) partially with respect to *t*, we get



Adding (3) and (5), we get





which is the required partial differential equation.

**Exercise: Try yourself:** Find the partial differential equation by eliminating constants from the following relations:

1. (*constants A, p* )
2. (*constants a , b* )
3. (*constants a , b, c* )
4. (*constants a , b* )

**Linear Partial Differential Equations of Order One:** A differential equation involving derivatives *p* and *q* only and no higher is called of order one. If, in addition, the degree or power of *p* and *q* is unity, then it is a linear partial differential equation of order one.

**Example: 1.**

**2.** 

The standard form of linear partial differential equation of order one is,



where, *P*, *Q* and *R* being functions of *x*, *y* and *z*. This is also known as Lagrange equation.

The general solution of (1) is,



where  is an arbitrary function and and are solutions of equations,



which are called Lagrange auxiliary or subsidiary equations for (A).

**Working procedure for solving by Lagrange’s method:**

**Step-1:** Put the given linear partial differential equation in the standard form,

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**Step-2:** Write down Lagrange’s auxiliary equations for (1) namely,



**Step-3:**Solve (2) by well-known methods. Let  and be two independent solutions of (2).

**Step-4:** The general solution (or integral) of (1) is then written in one of the following three equivalent forms:

 ,and.

**There are four types of problems based on :**

**Type-I:** If one of the variables is either absent or cancels out from any two fractions of given equations (*B*). Then an integral can be obtained by the usual methods. The same procedure can be repeated with another set of two fractions of the given equations (*B*).

**Problem-01:** Solve 

**Solution:** Given that, 

The Lagrange’s auxiliary equations for (1) are,



Taking the first two fractions of (2), we get









Integrating,





Next, taking the first and the last fractions of (2), we get









Integrating,





From (3) and (4) the required general solution (integral) is,



where,  is an arbitrary constant.

**Problem-02:** Solve 

**Solution:** Given that, 

The Lagrange’s auxiliary equations for (1) are,



Taking the first two fractions of (2), we get







Integrating,







Next, taking the last two fractions of (2), we get







Integrating,







From (3) and (4) the required general solution (integral) is,



where,  is an arbitrary constant.

**Problem-03:** Solve 

**Solution:** Given that, 

The Lagrange’s auxiliary equations for (1) are,



Taking the first and the last fractions of (2), we get







Integrating,





Next, taking the last two fractions of (2), we get





Integrating,



From (3) and (4) the required general solution (integral) is,



where,  is an arbitrary constant.

**Problem-04:** Solve 

**Solution:** Given that, 

The Lagrange’s auxiliary equations for (1) are,



Taking the first two fractions of (2), we get







Integrating,





Next, taking the last two fractions of (2), we get











which is a linear equation.



From (4), we get





Integrating,







From (3) and (4) the required general solution (integral) is,



where,  is an arbitrary constant.

**Exercise:**

1. 
2. 
3. 
4. 
5. 

**Problem-05:** Solve 

**Solution:** Given that, 

The Lagrange’s auxiliary equations for (1) are



Equations of (2) can be written as,











Taking first two fractions of (3) we get







Integrating we get







Next, taking last two fractions of (3) we get







Integrating we get







The general solution of (1) is,



where,  is an arbitrary constant.

**Type-II**: If one integral of (*B*) is known by using rule-I but another integral cannot be obtained. Then the integral known to us is used to find another integral.

**Problem-01:** Solve 

**Solution:** Given that, 

The Lagrange’s auxiliary equations for (1) are



Taking first two fractions of (2) we get





Integrating, we get



Next, taking the first and last fractions of (2) we get







Integrating we get





The general solution of (1) is,



where,  is an arbitrary constant.

**Problem-02:** Solve 

**Solution:** Given that, 

The Lagrange’s auxiliary equations for (1) are



Taking first two fractions of (2) we get







Integrating, we get







Next, taking the last two fractions of (2) we get











Integrating we get





The general solution of (1) is,



where,  is an arbitrary constant.

**Problem-03:** Solve 

**Solution:** Given that, 

The Lagrange’s auxiliary equations for (1) are



Taking first two fractions of (2) we get







Integrating, we get







Next, taking the first and last fractions of (2) we get









Integrating we get







The general solution of (1) is,



where,  is an arbitrary constant.

**Exercise:**

1. **Solve** 
2. **Solve** 
3. **Solve** 
4. **Solve** 
5. **Solve** 

**Type-III:** Ifbe functions of *x*, *y* and *z* or constants, which are called multipliers. Then, by a well-known principle of algebra, each fraction in (*B*) will be equal to,



Ifthen we know thatand which can be integrated easily. This procedure can be repeated to obtain another integral. If another integral cannot be obtained by this process then the rule-I can be applied.

**Problem-01:** Solve 

**Solution:** Given that, 

The Lagrange’s auxiliary equations for (1) are



Choosing *x*, *y*, *z* as multipliers, each fraction of (2) is equal to





Integrating, we get





Again, choosing *l*, *m*, *n* as multipliers, each fraction of (2) is equal to





Integrating, we get



The general solution of (1) is,



where,  is an arbitrary constant.

**Problem-02:** Solve 

**Solution:** Given that, 

The Lagrange’s auxiliary equations for (1) are



Choosing *x*, *y*, *z* as multipliers, each fraction of (2) is equal to





Integrating, we get





Again, taking last two fractions of (2) we get













Integrating, we get





The general solution of (1) is,



where,  is an arbitrary constant.

**Problem-03:** Solve 

**Solution:** Given that, 

The Lagrange’s auxiliary equations for (1) are



Taking last two fractions of (2) we get







Integrating, we get







Again, choosing *x*, *y*, *z* as multipliers, we get



From the last two of this, we get







Integrating, we get







The general solution of (1) is,



where,  is an arbitrary constant.

**Problem-04:** Solve 

**Solution:** Given that, 

The Lagrange’s auxiliary equations for (1) are



Choosing *x*, *y*, *z* as multipliers, each fraction of (2) is equal to





Integrating, we get





Again, choosing as multipliers, each fraction of (2) is equal to





Integrating, we get







The general solution of (1) is,



where,  is an arbitrary constant.

**Problem-05:** Solve 

**Solution:** Given that, 

The Lagrange’s auxiliary equations for (1) are



Choosing *x*, *y*, *z* as multipliers, each fraction of (2) is equal to





Integrating, we get





Again, choosing as multipliers, each fraction of (2) is equal to





Integrating, we get







The general solution of (1) is,



where,  is an arbitrary constant.

**Problem-06:** Solve 

**Solution:** Given that, 

The Lagrange’s auxiliary equations for (1) are



Choosing as multipliers, each fraction of (2) is equal to





Integrating, we get





Again, choosing as multipliers, each fraction of (2) is equal to







Integrating, we get



The general solution of (1) is,



where,  is an arbitrary constant.